Stability Analysis of a Fishery Model with Nonlinear Variation in Market price

Makwata Harun Omukuba
Department of Mathematics, Masinde Muliro University of Science and Technology, P. O. Box 190, Kakamega, Kenya
makwataharun@gmail.com

Adu A. M. Wasike
Department of Mathematics, Masinde Muliro University of Science and Technology, P. O. Box 190, Kakamega, Kenya
aduwasike@yahoo.com

Ochanda Jairus Bruno
Department of Education Communication Technology, Maseno University, Private Bag Maseno, Kenya
ochandajairusb@gmail.com

Abstract

A fishery model with price dependent harvesting is developed, by formulating a system of three differential equations describing the dynamics of the model. In order to reduce the dimensions of the system, it is aggregated to a system of two equations to ease analysis while preserving the necessary dynamics. Aggregation is attained by approximate aggregation where the system of three equations is replaced by a system of two equations through justified simplification in which evolution of the price is taken to evolve relatively faster than the evolution of the fish stock and the fishing effort. Local stability analysis is carried out and from the analysis, three cases of equilibrium solutions are obtained: unstable equilibrium, stable equilibrium and the co-existence of three strictly positive equilibria where two are stable, separated by a saddle. In the case of the co-existence of two stable equilibrium points, market prices are compared and a prediction of two kinds of fishery: an over-exploited fishery where the fishery supports a large economic activity but risks extinction and an under-exploited fishery where the stock is maintained at a large level far from extinction but the fishery only supports a small economic activity.

Key Words: Aggregation, Equilibrium Points, Local Stability

INTRODUCTION

Sustainable harvesting of renewable resources has been widely considered in bio-economics in an attempt to have sustainable resource exploitation (Clark, 1990; Cropper & Lee 1979; Walras, 1874). A fishery is such a renewable resource and fishery dynamics has been studied with earlier models considering population growth described by Verhulst‘s growth model

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right) \quad \ldots\ldots\ldots\ldots 1$$

where $n := n(t)$ represents the population of fish at time t, r the intrinsic growth rate and K is the carrying capacity (Murray, 1993).

When harvesting is considered, the model in equation 1 takes the form

$\frac{dn}{dt} = rn(1 - \frac{n}{K}) - h((n, E)$ where $h(n, E)$ is a function of $n \text{ and } E$ that is describing the fishing effort.

An appropriate choice of $h(n, E)$ is made mimicking predation depending on the fish resource and the fishing effort. Fishing effort, especially of an economic resource, is invariably dependent on the forces of demand and supply of the resource. For the case of a fishery, the demand and the price on the market are key to the investment in the fishing effort. Moreover, the demand will also determine the price and hence the effects of supply and demand need to be considered in a more realistic model. If we take supply and demand into consideration, a more feasible model to study the fishery dynamics with price dependent harvesting is

$\frac{dn}{dt} = f(n) - h(n, E)$ ………………..2(i)

$\frac{dE}{dt} = E(p, E, c)$ ……………………….2(ii)

$\frac{dp}{dt} = \propto p\{D(p) - S(p)\}$ ……………2(iii)

The first equation describes the evolution with time of the fish population, the second equation describes the evolution of the fishing effort, where $\frac{dE}{dt} = E(p, E, c)$ some is function dependent on the cost of harvesting c, and the benefit which depends on the market price p, such that the fishing effort is determined by the difference of the benefits and the cost of fishing. The last equation describes the evolution with time of the market price $p \coloneqq p(t)$ which depends on the supply and demand forces governed supply and demand functions $D(p) \text{ and } S(p)$. A choice of the demand function and a consideration of the price dynamics caused by constraints in harvesting and storage, determines the extent of the effort applied in the harvesting for the fishery to be of economical value. Smith (1969) considered a model with a constant price:

$$\begin{cases} \frac{dn}{dt} = f(n) - h(n, E), \\ \frac{dE}{dt} = \beta(ph(n, E) - cE). \end{cases} \quad \text{...3}$$

where $n \coloneqq n(t)$ represent the mass of the fish resource, while $E \coloneqq E(t)$ represents the fishing effort at time t. The function $f(n)$ is the natural growth function of the resource, $h(n, E)$ the harvesting function depending on the resource and the fishing effort. Smith (1969) considered a Holling Type II functional response as the harvesting function. The constant c is the cost per unit of fishing effort, β is an adjustment positive co-efficient depending on the fishery and p, the landed fish price per unit of the landed fish stock at time t. Stability analysis revealed two stable positive equilibria and a saddle. Barbier *et al.* (2002) proposed a time discrete version of model in equation 3, but used a Schaefer function as the functional response, that is

$h(n, E) = qn$ with q a positive constant called capturability (Schaefer, 1957). Stability analysis and pooled time analysis showed two stable fixed points and a saddle. Mchich *et al.* (2002) considered a model with linear variation in price given by:

$$\begin{cases} \frac{dn_1}{d\tau} = kn_2 - k'n_1 + \epsilon\{r_1 n_1(1 - \frac{n_1}{k_1}) - q_1 n_1 E_1\} \\ \frac{dn_2}{d\tau} = -kn_2 + k'n_1 + \epsilon\{r_2 n_2(1 - \frac{n_2}{k_2}) - q_2 n_2 E_2\} \end{cases} \quad \ldots\ldots\ldots\ldots 4$$

$$\begin{cases} \frac{dE_1}{d\tau} = mE_2 - m'E_1 + \epsilon E_1\{pq_1 n_1 - c_1\} \\ \frac{dE_2}{d\tau} = -mE_2 + m'E_1 + \epsilon E_2\{pq_2 n_2 - c_2\}, \end{cases}$$

Where, the differentiation is with respect to a fast time scale, a spatial fishery with two fishing areas is considered. Stability analysis reveals a stable equilibrium and a stable limit cycle. Auger *et al.* (2009) modified the model in Equation 4, by considering the effects of the market price on the dynamics of Equation 4. This was done by the addition of the equation;

$$\frac{dp}{d\tau} = \alpha\{D(p) - (q_1 n_1 E_1 + q_2 n_2 E_2)\} \quad \ldots\ldots\ldots\ldots 5$$

Where $p := P(t)$ is the price of the stock of fish at any time t, α a positive constant describing the speed of price adjustment on the market and $D(p)$ a demand function which is assumed to decrease with increasing price. This equation describes the variation of the price with time and it is assumed that the price varies according to the difference between the demand and the supply. Stability analysis done on this model reveals co-existence of two stable positive equilibrium with a separatrix associated to the stable manifold of the unstable equilibrium in the middle.

DISCUSSIONS

The Model

A simple time continuous model describing the relationship between the three main variables; Fish stock, harvesting effort and the market price alongside the parameters is presented in equation 6:

$$\begin{aligned} \dot{n} &= f(n) - h(n, E) \\ \dot{E} &= \beta(ph(n, E) - cE) \quad \ldots\ldots\ldots\ldots 6 \\ \dot{p} &= \alpha p(D(p) - h(n, E)) \end{aligned}$$

In the equation 6 $f(n) = rn(1 - \frac{n}{k})$ is the logistic growth rate function, the harvesting function $h(n, E)$ depending on the fish resource and the fishing effort mimicks predation and takes the form $h(n, E) = g(n, E)E$, where the function $g(n, E)$ is called the functional response, which is the amount of fish captured per unit of fishing. Schaefer function $g(n, E) = qn$ is chosen since it takes into account mortality and harvesting as control variables to the stock growth, q is a positive constant referred to as capturability. With this choice of $f(n)$ and $h(n, E)$, the first equation in 6, becomes $\dot{n} = rn(1 - \frac{n}{k}) - qnE$. The second equation in 6, describes the evolution of the fishing effort depending on the difference between the benefit and the cost of fishing. This dynamics of harvesting are described by; $\dot{E} \propto (benefit - \cos t)$, with the total benefit being the product of the market price and the total catch, while the total cost being the product of the cost per unit fishing and the fishing effort, thus we have $\dot{E} = \beta(ph(n, E) - cE)$, where β a constant of proportionality is is a positive adjustment coefficient. The third equation in 6, describes the variation of market price, which depends on the demand, supply of the fish and the price dynamics. Assuming that relative variation in the market price is governed by a simple balance between demand and supply of the fish which is simply the catch, this relation is represented by; $\frac{\dot{p}}{p} \propto D(p) - S(p)$, (Mackay, 1989). Taking α as a constant of proportionality that is referred to as the price adjustment parameter, thus the equation becomes, $\frac{\dot{p}}{p} = \alpha D(p) - S(p)$, with $D(p)$ and $S(p)$ denoting the demand and supply functions respectively. With the choice of $D(p) = A - p(t)$ where A is a positive constant parameter representing the limit threshold of the market price such that the demand decreases linearly with increasing market price, and $S(p) = qnE$, the equation in market price becomes; $\dot{p} = \alpha p(A - p - qnE)$. This equation has market price evolving nonlinearly depending on the price dynamics and the difference between demand and supply. The existence of the price dynamics in this equation is occasioned by the price fluctuations on the market forces caused by supply variations and reliability of storage facilities. This function makes Equation 6, to take the form;

$$\begin{aligned} \dot{n} &= rn(1 - \frac{n}{k}) - qnE, \\ \dot{E} &= \beta E(pqn - c), \\ \dot{p} &= \alpha p(A - p - qnE). \end{aligned} \qquad \ldots\ldots 7$$

Aggregated Model

Evolution of the market price is comparatively faster than the evolution of the fish stock and fishing effort. This is due to day to day variation of market price as suppliers adjust to market forces and fishery conditions in order to recoup their investment and make profit. Aggregation is achieved by replacing p, in the harvesting equation with its non trivial equilibrium values, $p := \overset{*}{p}$, which solves

$$\dot{p} = \alpha p(A - p - qnE) = 0 \quad \ldots\ldots 8$$

To obtain $\overset{*}{p} = A - qnE.$ ……9

Substituting Equation 9, in Equation 7, we obtain

$$\dot{E} = \beta E\{(A - qnE)qn - c\} \quad \ldots\ldots 10$$

If $\beta = 1$ in Equation 10, which is the maximum value in the range $0 \le \beta \le 1$ and may occur when the environmental conditions and harvesting are favorable for stock growth in the fishery. Thus a system of two differential equations below is obtained on aggregation.

$$\dot{n} = n\{r(1 - \frac{n}{k}) - qE\}$$
$$\dot{E} = E\{-c + qn(A - qnE)\} \quad \ldots\ldots 11$$

Equilibrium Points

This gives points where the dynamics of the system in equation 11, persists in time. The n nullclines are: $n = 0$, and $r(1 - \frac{n}{k}) - qE = 0$ while the E nullclines are $E = 0$ and $-c + qn(A - qnE) = 0.$ The equilibrium points are the intersections of E and n nullclines, that is

$E_0 = (0,0)$, $E_1 = (k,0)$ and $E_2 = (n^*, E^*)$ the solution (n^*, E^*) of

$$E(n) = \frac{r}{q}\left(1 - \frac{n}{k}\right),$$
$$E(n) = \frac{1}{qn}\left(A - \frac{c}{qn}\right) \qquad \text{……………………………………………………………..12}$$

Solution for c in Equation 12, gives a cubic equation for parameter c as a function of the equilibrium fish stock $c(n^*)$ given by;

$$c(n^*) = \frac{rq}{k} n^{*3} - rqn^{*2} + Aqn^* \qquad \text{……………………………………………….13}$$

Solution of equilibrium values of n^* in Equation 13, gives one or three values which satisfies the equilibrium conditions depending on k.

Local Stability Analysis

The system in Equation 11 is linearized about the equilibrium points and study the trace and determinant of the matrix of linearization as various parameters are varied. Equation 11, can be expressed as

$$f(n, E) = n\{r(1 - \frac{n}{k}) - qE\}$$
$$g(n, E) = E\{-c + qn(A - qnE)\} \qquad \text{……………………………………………….14}$$

The Jacobian matrix is

$$J(n, E) = \begin{pmatrix} f_n(n, E) & f_E(n, E) \\ g_n(n, E) & g_E(n, E) \end{pmatrix} = \begin{pmatrix} r - \frac{2nr}{k} - qE & -qn \\ qEA - 2q^2nE & -c + qnA - 2q^2n^2E \end{pmatrix}$$

At E_0, the Jacobian matrix is $J(0,0) = \begin{pmatrix} r & 0 \\ 0 & -c \end{pmatrix}$, whose eigenvalues are: r and $-c$. Since one is positive and the other is negative, this equilibrium point is a saddle.

At E_1, $J(k,0) = \begin{pmatrix} -r & -qk \\ 0 & -c + qkA \end{pmatrix}$. If $k < \frac{c}{Aq}$, both the eigenvalues $-r$ and $-c + qkA$ are negative and hence this equilibrium point is a stable equilibrium but if $k > \frac{c}{Aq}$, then E_1 is a saddle point. At the equilibrium E_2, the Jacobian matrix is given by, $J(n^*, E^*) = \begin{pmatrix} -\frac{r}{k}n^* & -qn^* \\ qE^*(A - 2qn^*E^*) & -q^2 n^{*2} E^* \end{pmatrix}$.

The trace and determinant of this matrix is $tr(J) = -\frac{r}{k}n^* - q^2 n^{*2} E^* < 0$ and determinant of $J(n^*, E^*) = \det(J) = q^2 n^* E^* (\frac{r}{k} n^{*2} + A - 2qn^* E^*)$. Using the first equation in 12, in $\det(J)$, we obtain

$\det(J) = q^2 n^* E^* \varphi(n^*)$ Where $\varphi(n^*) = \frac{3r}{k} n^{*2} - 2rn^* + A$. Since $q^2 n^* E^*$ is positive, the sign of $\det(J)$ will depend on $\varphi(n^*)$. Since $c'(n) = \varphi(n)$, where the prime indicates differentiation with respect to n, we study the sign of $c'(n^*)$ by analyzing Equation 13. For different values of $k = 2,3,4, and 5$ the number of zeroes vary. There are special points of $c(n^*)$ where two zeroes merge; this happens when $c'(n^*) = \frac{3rq}{k} n^{*2} - 2rqn^* + Aq = q\varphi(n^*) = 0.$ The solution n^* for $c'(n^*) = 0$ are $n^*_{1,2} = \frac{k}{3}\left(1 \pm \sqrt{1 - \frac{3A}{kr}}\right)$ ……………………………………….15

If $r < \frac{3A}{k}$, then $c'(n^*)$ is positive and $c(n^*)$ is monotonic increasing with complex roots. If $r > \frac{3A}{k}$, then there two real zeroes for $c'(n^*)$. If $r = \frac{3A}{k}$, the two real zeroes coincide. At this point, $n^*_{1,2} = \frac{k}{3}$, furthermore, $c = \frac{k}{9}$ and $E^* = \frac{r}{q}(1 - \frac{n^*}{k}) = \frac{2A}{k}$. Two different cases are distinguished in the following propositions and their proofs:

Proposition 1

For $0 < r < \frac{3A}{k}$ and $k > \frac{c}{Aq}$, E_1 is a saddle point and E_2 is a positive stable equilibrium point.

Proof. If $0 < r < \frac{k}{3A}$ in 15, the sign of $c^{'}(n^{*})$ which is the same as the sign of $\varphi(n^{*})$ does not change and is always positive. This implies that $\det(J) > 0$.

Moreover, $c^{''}(n^{*}) = \frac{6rq}{k} n^{*} - 2rq$ implies that $n^{*} = \frac{k}{3}$ is a point of inflection.

We have $c(k) = qAk$ but since C is strictly increasing and may take positive or negative values depending on k, we consider $c(k) = qAk - c$ and as $\lim_{n\to+\infty} c(n) = +\infty$, we conclude that C vanishes at a unique point n^{*}, thus we obtain a unique equilibrium point E_2. If $k < \frac{c}{Aq}$, then $c(k) < 0$ and C vanishes at a value $n^{*} > k$, which corresponds to a negative fishing effort equilibrium $(E^{*} < 0)$. In this case, the equilibrium point E_1 is a stable equilibrium but (n^{*}, E^{*}) does not present any interest since it is corresponding to unrealistic negative fishing effort, but if $k > \frac{c}{Aq}$ then $c(k) > 0$ and C vanishes at a value $n^{*} < k$, with a positive fishing effort equilibrium $E^{*} > 0$. In this case E_1 is a saddle point and E_2 is the unique positive stable equilibrium point as $tr(J(n^{*}, E^{*})) < 0$ and $\det(J(n^{*}, E^{*})) > 0$.

Proposition 2

For $0 < \frac{3A}{k} < r$, $E_i = (n_i^{*}, E_i^{*})$ for $i = 1,2,3$ are three positive equilibrium points such that we have the following cases:

1. If $c(n^{*}) < 0, c(n_1^{*}) < 0,$ we obtain a unique positive and stable equilibrium point (n^{*}, E^{*});

2. If $c(n_1^*) > 0$ and $c(n_2^*) < 0,$ we obtain a unique positive and stable equilibrium point (n^*, E^*);

3. If $c(n_1^*) > 0$ and $c(n_2^*) < 0$, we obtain three positive equilibrium points $c(n_i^*, E_i^*)$ for $i = 1,2,3$ whereby (n_1^*, E_1^*) and (n_3^*, E_3^*) are stable while (n_2^*, E_2^*) is saddle equilibrium point.

Proof. If $0 < \frac{3A}{k} < r$ in Equation 15, c' vanishes at two values n_1 and n_2 given by $0 \le n_1 = \frac{k}{3}(1 - \sqrt{1 - \frac{3A}{rk}}) < \frac{k}{3},$ and $\frac{k}{3} < n_2 = \frac{k}{3}(1 + \sqrt{1 - \frac{3A}{rk}}) < k.$ since $\det(J)$, $\varphi(n)$ and $c'(n)$ have the same sign, we have: $\det(J) > 0,$ if $n \in [0, n_1) \cup (n_2, +\infty],$ $\det(J) < 0$ if $n \in (n_1, n_2)$.

Recall that $\lim_{n^* \to +\infty} c(n^*) = +\infty.$ As $c(n_1^*)$ and $c(n_2^*)$ can have positive or negative signs, so for case 1, with $c(n^*) < 0,$ $c(n_1^*) < 0$ and $n^* > n_1$, $\det(J) > 0$ and $tr(J) < 0$ thus a stable equilibrium point (n^*, E^*). For case 2, since $c(n_1^*) > 0$ and $c(n_2^*) > 0,$ with $n^* < n_1^*$, $\det(J) > 0$ and $tr(J) < 0$ thus a stable equilibrium point (n^*, E^*). Finally for case 3, given that $n_1^* < n_1 < n_2^* < n_2 < n_3^*$ is satisfied, (n_1^*, E_1^*) and (n_3^*, E_3^*) are stable since and $tr(J)9 < 0$ while (n_2^*, E_2^*) is a saddle point since $\det(J) < 0$ and $tr(J) < 0$.

Comparison of the Fish Price in the Case of Two Stable Positive Equilibria

If $r > \frac{3A}{k}$, the three equilibria $(n_1^*, E_1^*), (n_2^*, E_2^*), (n_3^*, E_3^*)$ are in the positive quadrant with (n_2^*, E_2^*) being a saddle while the other two being stable equilibria.

Assume $n_3^* > n_1^*$, we have $E_1^* = \frac{1}{qn_1^*}\left(A - \frac{c}{qn_1^*}\right)$, $E_3^* = \frac{1}{qn_3^*}\left(A - \frac{c}{qn_3^*}\right)$ and $p_1^* = A - qn_1^* E_1^*$, $p_3^* = A - qn_3^* E_3^*$.

Combining these sets of equations above, we obtain $p_1^* - p_3^* = \frac{c}{q}\left(\frac{n_3^* - n_1^*}{n_1^* n_3^*}\right)$. The sign of the difference of price at equilibrium is opposite to the difference of the fish population. Thus, if $n_3^* > n_1^*$, then we have $p_1^* > p_3^*$. In general we have the following set of inequalities: $n_3^* > n_1^*$, $E_3^* < E_1^*$ and $p_3^* < p_1^*$. Explained as, at equilibrium, the larger the fish stock the smaller the fishing effort and smaller is the market price.

CONCLUSION

The stability analysis of this model shows that depending on the parameter values of k and c, one, two or three strictly positive equilibria can occur. Very Unstable state $E_0 = (0,0)$, which denotes absence of fish population or if there is an introduction of the fish population, the population increases due to natural growth which definitely attracts fishing activity .Whereas, if the fish population is diminishing due to environmental conditions or over-exploitation, the fishing effort will also decline. $E_1 = (k,0)$ a stable equilibrium where the fish population is maintained at its carrying capacity and with absence of harvesting , the fishery persists at the carrying capacity. And the co-existence of two strictly positive equilibria $E_2 = (n^*, E^*)$ which corresponds to an over-exploited fishery (n_1^*, E_1^*) permitting large fishing effort and an economic activity with a satisfying market price p^*, and an under - exploited fishery (n_3^*, E_3^*) that maintains the fish stock at a desirable large level far from extinction but does not support any important economic activity.

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BIO-DATA

Makwata Harun Omukuba (Mr.) is currently a lecturer in the Department of Mathematics-Masinde Muliro University of Science and Technology. He is also a PhD. Candidate in Applied Mathematics, specializing in Dynamical Systems.